REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
10-12-2010	Article	DEC 2010 - JAN 2011

4. TITLE AND SUBTITLE
Multicast Queueing Delay: Performance Limits and Order-Optimality of Random Linear Coding

5a. CONTRACT NUMBER
FA8720-05-C-0002

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Brooke Shrader and Randy Cogill

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
MIT Lincoln Laboratory
244 Wood Street
Lexington, MA 02420

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
DDRE, 1777 N. Kent St.
Rosslyn, VA 22209

10. SPONSOR/MONITOR'S ACRONYM(S)
OSD-ATL

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
DISTRIBUTION STATEMENT A. Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
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15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: U			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					Zach Sweet
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER *(include area code)*
U	U	U	SAR	9	781-981-5997

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

Multicast Queueing Delay: Performance Limits and Order-Optimality of Random Linear Coding

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***Abstract*—In this work we analyze the average queue backlog for transmission of a single multicast flow consisting of M destination nodes in a wireless network. In the model we consider, the channel between every pair of nodes is an independent identically distributed packet erasure channel. We first develop a lower bound on the average queue backlog achievable by any transmission strategy; for a single-hop multicast transmission, our bound indicates that the queue size must scale as at least $\Omega(\ln(M))$. Next, we generalize this result to a multihop network and obtain a lower bound on the queue backlog as it relates to the minimum-cut capacity of the network. We then analyze the queue backlog for a strategy in which random linear coding is performed over groups of packets in the queue at the source node of a single-hop multicast. We develop an upper bound on the average queue backlog for the packet-coding strategy to show that the queue size for this strategy scales as $O(\ln(M))$. Our results demonstrate that in terms of the queue backlog for single-hop multicast, the packet coding strategy is order-optimal with respect to the number of receivers.**

I. Introduction

In this paper we analyze the size of the queue backlog for a single multicast flow transmitted over a network with links subject to packet erasures. A primary focus of our work is in establishing performance limits that characterize the minimum achievable expected queue length in the network. We also provide a closed-form upper bound on the expected queue length when a random linear coding scheme is applied for a one-hop multicast flow.

The throughput for multicast transmissions in a wireless network and the role of coding schemes in achieving good performance has been addressed extensively in the past decade. Random linear coding of packets in a multicast flow was first introduced in [11], which provides a lower bound on the probability that all multicast receivers are able to successfully decode the message sent by the source and shows that this scheme can outperform a randomized routing scheme. The achievable multicast throughput in a wireless erasure network is characterized with a minimum-cut representation in [9] and linear network coding is shown to be sufficient for achieving this result.

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This work appeared in part at the 2009 Allerton Conference on Communication, Control and Computing [6].

This work was sponsored by the Department of Defense under Air Force Contract FA8721-05-C-0002. Opinions, interpretations, conclusions, and recommendations are those of the author and are not necessarily endorsed by the United States Government.

While the delay performance of coding schemes has received less attention than the throughput performance, significant progress has been made in previous work. The delay or completion-time performance for one-hop multicast with link erasures was first characterized in [10], which shows that a block-based random linear coding scheme offers significant delay improvements over packet scheduling strategies. The work in [10] does not address queueing delay, but numerous subsequent works do. In [12], the authors consider a network with finite memory or buffers for storing packets and a scheme for coding over packets that arrive through a random process for a single unicast flow. They provide a framework for this problem that allows for computation of the delay and queue blocking/loss probability and demonstrate that packet coding schemes can offer good throughput performance under the constraint of limited memory for storing packets at intermediate nodes. A bulk-service queueing model for random linear coding over a unicast flow is developed in [13] and numerical results on the queueing delay demonstrate that by adapting the code block size to the number of buffered packets, coding schemes can offer good delay performance when the arrival rate is small. The work in [7] provides analytical bounds on the completion time and stable throughput for random linear coding across multiple multicast flows; the results indicate that although coding across flows requires unintended recipients to decode packets, the coding strategy can provide larger throughput than uncoded transmission. In contrast to the block-based coding schemes considered in the works above, [14] introduces an online or window-based coding scheme and proposes a new packet acknowledgment strategy for random linear coding based on acknowledging degrees of freedom received. The authors also provide a queue-length analysis for their policy and the queue size is shown to grow more slowly with load factor than a baseline acknowledgment strategy. Finally, while most works cited above address delay performance for intra-flow coding, [1] considers coding across flows and shows that significant delay penalties are incurred for synchronized coding across flows, but asynchronous coding of packets across flows can reduce the queueing delay.

Our work focuses on the multicast problem and differs from previous work in the following ways. First, we provide a lower bound on the minimum achievable queue length for any packet transmission strategy. For the one-hop multicast problem, our lower bound is a closed-form expression that provides insight on the best-case behavior of the queue length as a function of the number of multicast receivers and the link loss probability. For multicast in a general erasure network, our lower bound relates the minimum achievable queue length to the minimum-cut capacity of the network. Lastly, our upper bound on the queue length for a packet coding strategy allows us to conclude that for the one-hop multicast problem, random linear coding is order-optimal with respect to the number of multicast receivers.

II. The Problem

We consider the problem of multicast transmission of data packets from a single transmitter to a group of M receivers. We model a single multicast flow in which packets arrive at the transmitter or source node according to a Bernoulli process with rate λ. At other nodes in the network, a transmitted packet is received with some fixed probability, independent of other receivers and of past receptions. Our goal is to characterize the expected number of packets queued in the network in the steady state, and to devise a transmission scheme that minimizes this average queue backlog, thereby minimizing the queueing delay.

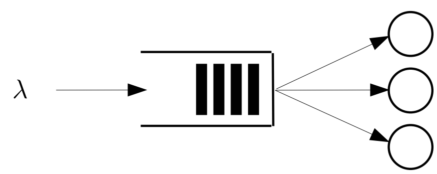

Fig. 1. Queueing system for one-hop multicast.

Most of our work in this paper focuses on the one-hop multicast problem, in which each of the M receivers is connected to the source by a single link, as shown in Figure 1. In this problem, there is a single queue of packets and the transmitter makes a transmission in every time slot. Each receiver receives a transmitted packet with probability q. Even for the simple one-hop multicast problem, minimizing expected queue length is deceptively difficult. To see why this is the case, first consider the simple scheme that retransmits the head of line packet until it has been received by all M receivers. As shown in [7], the expected number of time slots that a packet stays at the head of the queue is greater than or equal to

$$\frac{\ln(M+1)+0.3}{-\ln(1-q)}. \tag{1}$$

Therefore, this scheme cannot stabilize arrival rates satisfying

$$\lambda > \frac{-\ln(1-q)}{\ln(M+1)+0.3}. \tag{2}$$

This means that the expected queue length becomes arbitrarily large for arrival rates approaching some value greater than the right-hand side of (2). In other words, for any λ, one can choose M sufficiently large so that the expected queue length under the retransmission strategy is arbitrarily far from optimal.

We could instead approach the problem of minimizing expected queue length as a purely control theoretic problem. Rather than simply transmitting the head of line packet, in each time slot any packet in the queue can be transmitted. When the state of each channel connecting the transmitter to each receiver is known before transmitting a packet, a controller would use this information, together with the reception history of each packet in the queue to decide which packet to send next. The complexity and information requirements of such a scheme are clearly very high. On the other hand, without channel state information, throughput of such a scheme is no better than the throughput of the simple retransmission scheme. To see why this is true, if the system is ignorant of the channel states before transmitting a packet, the expected number of times a packet must be transmitted before successful reception still must be greater than or equal to (1), even if transmissions are attempted out of order.

For the one-hop multicast problem, it is well known that the queue can be stabilized for all arrival rates satisfying $\lambda < q$ using simple random linear coding schemes (see [8], for example). These schemes operate by collecting large blocks of packets in the queue, then transmitting encoded packets formed from this block until all receivers can decode all packets in the block. To stabilize rates λ approaching q, arbitrarily large blocks of packets must be formed. The block coding operation must create a backlog that grows with the size of the block. So, it seems that schemes that code over fixed-length blocks of packets are not well suited for the problem of minimizing expected backlog.

III. Main Results

In this paper we establish the following results about the expected steady state queue length for multicast.

- We show that for the one-hop multicast problem, the expected steady-state queue length of any strategy must satisfy
$$\liminf_{t\to\infty} \mathbf{E}[Q(t)] \geq \frac{3}{4}\left(\frac{\lambda\ln(M)-1}{-\ln(1-q)}\right),$$
where $Q(t)$ is the number of packets in the source queue at time t. This is true even for strategies that exploit channel state information.
- Using the same techniques, we establish a lower bound on the expected steady-state queue length for a general erasure network, where multicast receivers may be located multiple hops away from the source and packets may be queued at intermediate nodes in the network. This lower bound relates the queue length to the minimum-cut capacity of the network.
- For the one-hop multicast problem, we show the queue length process of a simple random linear coding scheme (at packet departure times) satisfies
$$\limsup_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \mathbf{E}[Q(t_k)] \leq$$
$$4\ln(M)\left(\frac{\lambda}{q-\lambda}\right)^2 + \left(8\sqrt{\ln(M)}+6\right)\frac{\lambda}{q-\lambda}.$$
Here, t_n is the time at which the n-th block of packets departs the system.

So, if the queue length process at departure times corresponds to the true queue length process, then for the one-hop multicast problem, the expected queue length of the random linear coding strategy is order optimal with respect to the number of receivers.

A. Lower bounds on achievable backlog for one-hop multicast

Here we present a lower bound on the minimum achievable steady state expected queue length for one-hop multicast. Specifically, we show that backlog must scale at least logarithmically with M, the number of receivers. In Section III-C, we show that the backlog of the code over queue contents strategy scales logarithmically with M. This implies that coding over the queue contents is order-optimal with respect to the number of receivers for one-hop multicast. The theorem that will be proved in this section is the following:

Theorem 1: Under any strategy

$$\liminf_{t\to\infty} \mathbf{E}[Q(t)] \geq \frac{3}{4}\left(\frac{\lambda \ln(M) - 1}{-\ln(1-q)}\right).$$

The proof of this theorem is at the end of this section, after several supporting lemmas are established.

The lower bound presented in this section makes very few assumptions on the strategy used. We will start by stating these assumptions in words, then give a more precise condition that must be satisfied by all policies.

To make the motivation for our assumptions clear, first consider a system composed of a single transmitter and a single receiver. Packets arrive at the transmitter according to a Bernoulli process with rate λ. In each time slot the transmitter is connected to the receiver with probability q, independent of past connections.

Various strategies could be used to transmit packets to the receiver. Regardless of the strategy used, we always assume that the system possesses several properties:

(1) At any time, the total number of packets that have been removed from the queue does not exceed the total number of packets that have entered the queue.
(2) At any time, the total number of packets that have been removed from the queue does not exceed the total number of time slots where the transmitter has been connected to the receiver. So, even if coding is applied, we do not consider schemes that compress packets. We can only transmit m packets to the receiver if the transmitter and receiver have been connected in at least m time slots.
(3) Connections cannot be used to transmit future arrivals. In other words, the strategy must be *causal*. We cannot send information about a packet that has not yet arrived in the queue.
(4) The queue starts out empty. Since we are concerned with steady-state queue length, this is without loss of generality.

To make these conditions precise, we'll introduce some notation. Let $Q(t)$ be the length of the source queue at time t. Let $A(t)$ be the random variable with $A(t) = 1$ if there is an arrival at time slot t and $A(t) = 0$ otherwise. Let $S(t)$ be the random variable with $S(t) = 1$ if the receiver is connected at time slot t and $S(t) = 0$ otherwise. Finally, let $D(t)$ be the number of departures from the queue at time t.

In terms of A, D, and S, properties (1), (2), and (3) are captured by the following condition. For all $\tau \leq t$, we require that D satisfies

$$\sum_{k=0}^{t-1} D(k) \leq \sum_{k=0}^{\tau-1} A(k) + \sum_{k=\tau}^{t-1} S(k). \tag{3}$$

Properties (1) and (2) imply this condition for $\tau = t$ and $\tau = 0$, respectively. Showing that property (3) implies this condition for all other τ is a little less obvious. To show this, consider the set of packets that arrived in the system during the interval $[0, t-1]$. We can partition this set into two sets $\mathcal{A}_1$ and $\mathcal{A}_2$, where $\mathcal{A}_1$ is the set of packets arriving in the interval $[0, \tau-1]$ and $\mathcal{A}_2$ is the set of packets arriving in the interval $[\tau, t-1]$. The total number of departures in the interval $[0, t-1]$ equals the total number of packets from $\mathcal{A}_1$ that depart the system in the interval $[0, t-1]$ plus the total number of packets from $\mathcal{A}_2$ that depart the system in the interval $[0, t-1]$. The total number of departures from $\mathcal{A}_1$ is, by property (1), less than or equal to the total number of packets in $\mathcal{A}_1$, which is

$$\sum_{k=0}^{\tau-1} A(k).$$

Since all packets in $\mathcal{A}_2$ arrived on or after τ, property (3) implies that no connection prior to τ can be used to serve a packet in $\mathcal{A}_2$. This, together with property (2), implies that the total number of departures from $\mathcal{A}_2$ is less than or equal to the total number of connections in the interval $[\tau, t-1]$, which is

$$\sum_{k=\tau}^{t-1} S(k).$$

Finally, summing these upper bounds on the number of departures from $\mathcal{A}_1$ and $\mathcal{A}_2$ gives (3).

Using property (3), we have the following lemma. This Lemma is very similar to a well known result that obtains a queue length process by applying a reflection mapping to a netput process (see Proposition 6.3 in [2], for example).

Lemma 1: If $Q(0) = 0$, the queue length at time t satisfies

$$Q(t) \geq \max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t-1} \big(A(k) - S(k)\big) \right\}$$

for any service policy with departures satisfying (3). Moreover, this inequality is tight when packets depart the queue as services occur.

Proof: At any time t,

$$Q(t) = \sum_{k=0}^{t-1} \big(A(k) - D(k)\big)$$

Since
$$\sum_{k=0}^{t-1} D(k) \leq \sum_{k=0}^{\tau-1} A(k) + \sum_{k=\tau}^{t-1} S(k)$$
for all $\tau \in [0,t]$, the queue length satisfies
$$\begin{aligned} Q(t) &\geq \sum_{k=0}^{t-1} A(k) - \sum_{k=0}^{\tau-1} A(k) - \sum_{k=\tau}^{t-1} S(k) \\ &= \sum_{k=\tau}^{t-1} \big(A(k) - S(k)\big) \end{aligned}$$
for all $\tau \in [0,t]$. Since this holds for all τ, clearly
$$Q(t) \geq \max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t-1} \big(A(k) - S(k)\big) \right\}$$

The fact that the inequality is tight if packets depart the queue as services occur can be shown by induction. Since we start with $Q(0) = 0$,
$$\begin{aligned} Q(1) &= \max\{0, A(0) - S(0)\} \\ &= \max_{1 \geq \tau} \left\{ \sum_{k=\tau}^{0} \big(A(k) - S(k)\big) \right\}. \end{aligned}$$
Now suppose $Q(t)$ satisfies
$$Q(t) = \max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t-1} \big(A(k) - S(k)\big) \right\}.$$
Then the queue length at time $t+1$ is
$$\begin{aligned} Q(t+1) &= \max\{0, Q(t) + A(t) - S(t)\} \\ &= \max\left\{0, \ \max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t} \big(A(k) - S(k)\big) \right\}\right\} \\ &= \max_{t+1 \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t} \big(A(k) - S(k)\big) \right\}. \end{aligned}$$
■

We will now extend these results to the multicast case. Let $S_j(t)$ be the random variable with $S_j(t) = 1$ if the transmitter is connected to the j-th receiver at time slot t, and $S_j(t) = 0$ otherwise. Let $R_j(t)$ be the number of packets received by receiver j at time t. Each $R_j(t)$ must satisfy the property (3). That is, for all $\tau \leq t$, the total number of packets received by receiver j up to time $t-1$ satisfies
$$\sum_{k=0}^{t-1} R_j(k) \leq \sum_{k=0}^{\tau-1} A(k) + \sum_{k=\tau}^{t-1} S_j(k).$$
Also, since packets do not leave the queue until they have been received by all receivers, the total number of departures in the interval $[0, t-1]$ satisfies
$$\sum_{k=0}^{t-1} D(k) \leq \min_j \left\{ \sum_{k=0}^{t-1} R_j(k) \right\}$$
Under this condition alone, we can establish a lower bound on the achievable expected queue backlog. The next lemma gives a lower bound in terms of the expected value of the minimum of binomial random variables.

Lemma 2: Let $Q(t)$ be the queue length of the multicast system at time t. Under any policy,
$$\liminf_{t \to \infty} \mathbf{E}[Q(t)] \geq \sup_{n \geq 0} \{\lambda n - f_M(n)\},$$
where $f_M(n)$ is the expected minimum of M independent binomial random variables, each with parameters (q, n).

Proof: The total number of packets in the queue at time t is
$$Q(t) = \sum_{k=0}^{t-1} A(k) - \sum_{k=0}^{t-1} D(k).$$
Note that for all $t \geq \tau \geq 0$,
$$\begin{aligned} \sum_{k=0}^{t-1} D(k) &\leq \min_j \left\{ \sum_{k=0}^{t-1} R_j(k) \right\} \\ &\leq \sum_{k=0}^{\tau-1} A(k) + \min_j \left\{ \sum_{k=\tau}^{t-1} S_j(k) \right\}. \end{aligned}$$
By Lemma 1, $Q(t)$ satisfies
$$Q(t) \geq \max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t-1} A(k) - \min_j \left\{ \sum_{k=\tau}^{t-1} S_j(k) \right\} \right\}.$$
By Jensen's inequality,
$$\begin{aligned} &\mathbf{E}[Q(t)] \\ &\geq \mathbf{E}\left[\max_{t \geq \tau \geq 0} \left\{ \sum_{k=\tau}^{t-1} A(k) - \min_j \left\{ \sum_{k=\tau}^{t-1} S_j(k) \right\} \right\} \right] \\ &\geq \max_{t \geq \tau \geq 0} \left\{ \mathbf{E}\left[\sum_{k=\tau}^{t-1} A(k) - \min_j \left\{ \sum_{k=\tau}^{t-1} S_j(k) \right\} \right] \right\} \\ &= \max_{t \geq \tau \geq 0} \{\lambda(t-\tau) - f_M(t-\tau)\} \\ &= \max_{t \geq n \geq 0} \{\lambda n - f_M(n)\} \end{aligned}$$
where $f_M(n)$ is the expected minimum of M independent binomial random variables, each with parameters (q, n). Finally,
$$\begin{aligned} \liminf_{t \to \infty} \mathbf{E}[Q(t)] &\geq \lim_{t \to \infty} \left(\max_{t \geq n \geq 0} \{\lambda n - f_M(n)\} \right) \\ &= \sup_{n \geq 0} \{\lambda n - f_M(n)\}. \end{aligned}$$
■

The following Lemma is used in the proof of Theorem 1.

Lemma 3: For all $x \geq a > 0$,
$$\left(1 - \frac{a}{x}\right)^x \leq e^{-a}$$

Proof: It is well known that

$$\lim_{x\to\infty}\left(1-\frac{a}{x}\right)^x = e^{-a}.$$

In the remainder of the proof, we will show that $\left(1-\frac{a}{x}\right)^x$ is monotonically increasing for $x \geq a$.

Taking the derivative yields

$$\frac{d}{dx}\left(1-\frac{a}{x}\right)^x = \left(1-\frac{a}{x}\right)^x\left(\ln\left(1-\frac{a}{x}\right)+\frac{a}{x-a}\right).$$

The logarithmic part can be rewritten as

$$\begin{aligned}\ln\left(1-\frac{a}{x}\right) &= -\ln\left(\frac{x}{x-a}\right)\\ &= -\ln\left(\frac{x-a+a}{x-a}\right)\\ &= -\ln\left(1+\frac{a}{x-a}\right).\end{aligned}$$

For all $z \geq 0$, clearly $z \geq \ln(1+z)$. So,

$$\frac{a}{x-a}-\ln\left(1+\frac{a}{x-a}\right)\geq 0$$

for all $x \geq a$. Also,

$$\left(1-\frac{a}{x}\right)^x \geq 0$$

for all $x \geq a$. So,

$$\frac{d}{dx}\left(1-\frac{a}{x}\right)^x \geq 0$$

for all $x \geq a$. ■

Now we are ready to prove Theorem 1.

Proof of Theorem 1: By Lemma 2, for any $n \geq 0$,

$$\liminf_{t\to\infty}\mathbf{E}[Q(t)] \geq \lambda n - f_M(n),$$

where

$$f_M(n) = \mathbf{E}[\min\{X_1,\ldots,X_M\}],$$

is the expected minimum of M independent binomial random variables $X_1,\ldots,X_M$ each with parameters (q,n).

Let

$$\widehat{n}(M) = \frac{3}{4}\left(\frac{\ln(M)}{-\ln(1-q)}\right).$$

The proof will proceed by showing that

$$f_M(\widehat{n}(M)) \leq \frac{3}{4}\left(\frac{1}{-\ln(1-q)}\right) \quad (4)$$

for all $M \geq 1$. Therefore, for all $M \geq 1$ we will have the lower bound

$$\lambda\widehat{n}(M) - f_M(\widehat{n}(M)) \geq \frac{3}{4}\left(\frac{\lambda\ln(M)-1}{-\ln(1-q)}\right).$$

Note that the largest value taken by any (q,n) binomial random variable is n. To show (4), we will use the upper bound

$$\begin{aligned}\mathbf{E}[\min\{X_1,\ldots,X_M\}] &= \sum_{x=1}^{n} x\mathbf{P}(\min\{X_1,\ldots,X_M\}=x)\\ &\leq \sum_{x=1}^{n} n\mathbf{P}(\min\{X_1,\ldots,X_M\}=x)\\ &= n\mathbf{P}(\min\{X_1,\ldots,X_M\}>0).\end{aligned}$$

It is the case that $\min\{X_1,\ldots,X_M\}>0$ if and only if $X_i > 0$ for all i. The event $X_i > 0$ occurs with probability $1-(1-q)^n$. Since $X_1,\ldots,X_M$ are independent,

$$\begin{aligned}&\mathbf{P}(\min\{X_1,\ldots,X_M\}>0)\\ &\quad= \mathbf{P}(X_1>0)\cdots\mathbf{P}(X_M>0)\\ &\quad= \left(1-(1-q)^n\right)^M\\ &\quad= \left(1-e^{n\ln(1-q)}\right)^M.\end{aligned}$$

Using $n = \widehat{n}(M)$ gives

$$\begin{aligned}1-e^{\widehat{n}(M)\ln(1-q)} &= 1-e^{-(3/4)\ln(M)}\\ &= 1-M^{-3/4}\\ &= 1-\frac{M^{1/4}}{M}.\end{aligned}$$

Since $M^{1/4} \leq M$ for $M \geq 1$, we can use Lemma 3 to obtain

$$\begin{aligned}\mathbf{P}(\min\{X_1,\ldots,X_M\}>0) &= \left(1-\frac{M^{1/4}}{M}\right)^M\\ &\leq e^{-M^{1/4}}.\end{aligned}$$

To bound $f_M(\widehat{n}(M))$,

$$\begin{aligned}f_M(\widehat{n}(M)) &\leq \widehat{n}(M)\mathbf{P}(\min\{X_1,\ldots,X_M\}>0)\\ &\leq \frac{3}{4}\left(\frac{\ln(M)}{-\ln(1-q)}\right)e^{-M^{1/4}}\\ &= \frac{3}{4}\left(\frac{1}{-\ln(1-q)}\right)\ln(M)e^{-M^{1/4}}.\end{aligned}$$

Note that

$$\begin{aligned}\ln(M)e^{-M^{1/4}} &= e^{\ln(\ln(M))}e^{-M^{1/4}}\\ &= e^{\left(\ln(\ln(M))-M^{1/4}\right)}\end{aligned}$$

Since $x^{1/2} \geq \ln(x)$ for $x \geq 0$ and both $x^{1/2}$ and $\ln(x)$ are monotonically increasing for $x \geq 0$, $M^{1/4} \geq \ln(\ln(M))$ for all $M \geq 1$. Therefore, $\ln(\ln(M)) - M^{1/4} \leq 0$ for all $M \geq 1$, or equivalently $\ln(M)e^{-M^{1/4}} \leq 1$ for all $M \geq 1$. So,

$$f_M(\widehat{n}(M)) \leq \frac{3}{4}\left(\frac{1}{-\ln(1-q)}\right)$$

for all $M \geq 1$, giving the bound

$$\begin{aligned}\liminf_{t\to\infty}\mathbf{E}[Q(t)] &\geq \lambda\widehat{n}(M) - f_M(\widehat{n}(M))\\ &\geq \frac{3}{4}\left(\frac{\lambda\ln(M)-1}{-\ln(1-q)}\right).\end{aligned}$$

■

Another interpretation of the bound presented here relates to the response time of a parallel fork-join queue as described in [3]. The fork-join queue is a model employed for parallel processing systems in which an arriving job or customer is directed to M parallel, independent servers and service completion can only take place when all M servers have completed the task. In [3] results for a continuous-time fork-join queue are presented and it is shown that the average response time (or waiting time in the queue) is $O(\ln(M))$. The multicast problem we consider in this work might also be modeled as a fork-join queue in which each parallel server represents the process of transmission to one of the M destination nodes. By Little's Law, the average queue length is equal to λ times the average response time, so our result that the average queue length scales logarithmically with M is supported by the previous results on fork-join queues.

B. Lower bounds on achievable backlog for general erasure networks

To provide further insight on our bound on achievable backlog for the one-hop multicast problem, we next develop similar results for the generalized problem of an *erasure network*. This network is described by an acyclic directed graph $G = (V, E)$ with source vertex s and M terminal vertices $t_1, \ldots, t_M$. Time is slotted and in each time slot, edge $(i, j) \in E$ is connected with probability q_{ij}, independent of other edges and past edge connections. An attempted packet transmission from node i to node j is successful in a given time slot if edge (i, j) is connected. In the model we consider here, data packets are produced at the source vertex according to a Bernoulli process with rate λ, and each packet must be multicast to all terminal vertices. This model incorporates the broadcast nature of wireless networks, since packets transmitted by a node are heard by all neighboring nodes. However, this model assumes that multiple, distinct packets might be received by a node in a given time slot. Also, here we assume all successful packet transmissions are acknowledged by the receiving nodes, and acknowledgments are sent error-free.

Due to time-varying link connectivity and contention for link access, packets may be queued at multiple network nodes. Various strategies can be used for routing packets through the network, scheduling link access at the nodes, and possibly coding among packets in the network. Under any possible strategy, what is the smallest expected queue backlog we can accumulate at all network nodes? Here we outline a framework for analyzing this limit of achievable queue backlog in erasure networks.

The characterization of erasure network capacity considers the capacity across any *cut* in the network. A cut is a partition of the vertex set of the network into two complimentary subsets, denoted as V_c and $\overline{V_c}$, where $s \in V_c$ and $t_i \in \overline{V_c}$ for some terminal vertex t_i. We let $\mathcal{C}$ denote the set of all cuts in the network. The set of edges contained in a cut is $E_c = \{(i, j) \in E | i \in V_c, j \in \overline{V_c}\}$. Finally, we let the *frontier* of the cut be the set of vertices

$$F_c = \{\ i \in V_c \mid (i, j) \in E_c \text{ for some } j \in \overline{V_c}\ \}.$$

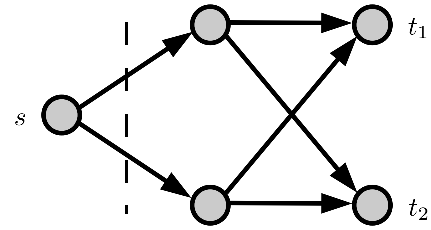

Fig. 2. Simple multicast network and a cut separating the source from each terminal.

The capacity of the cut c is given by

$$\sum_{i \in F_c} \left(1 - \prod_{j:(i,j) \in E_c} (1 - q_{ij}) \right)$$

This gives the maximum rate which packets can be transmitted across the cut. At best, packets can be transmitted simultaneously from each vertex on the frontier, leading to the sum in this expression. For each vertex in the frontier, only one edge needs to be connected to transmit a packet in a given time slot, leading to the product in this expression. Since all packets traveling from the source to some terminal must cross this cut, the capacity of the cut gives an upper bound on the source rates that can be stably supported by the network. It is also known that the minimum capacity over all cuts is exactly the maximum source rate that can be stably supported [9].

Let $A(k)$ and $D(k)$ denote the number of packets entering and departing the network in time slot k. The total number of packets queued in the network at time slot t is

$$Q(t) = \sum_{k=0}^{t-1} A(k) - \sum_{k=0}^{t-1} D(k).$$

Let $S_{(i,j)}(t)$ be a random variable with $S_{(i,j)}(t) = 1$ if edge (i, j) is connected in time slot t, and $S_{(i,j)}(t) = 0$ otherwise. We can bound the total number of departures in any interval by considering the connectivity of links crossing any cut. For any cut c and any times $t_1 \leq t_2$,

$$\sum_{k=0}^{t_2-1} D(k) \leq \sum_{k=0}^{t_1-1} A(k) + \sum_{k=t_1}^{t_2-1} \sum_{i \in F_c} \left(1 - \prod_{j:(i,j) \in E_c} \left(1 - S_{(i,j)}(k)\right) \right) \tag{5}$$

To understand where this bound comes from, suppose we start with $Q(0) = 0$. The total number of departures in the interval $[0, t_1 - 1]$ cannot be any greater than the total number of arrivals in this interval. Of the remaining packets arriving in the interval $[t_1, t_2 - 1]$, the total number of these packets departing the network cannot be greater than the total number of connections crossing any cut in the interval $[t_1, t_2 - 1]$. Summing these bounds for the intervals $[0, t_1 - 1]$ and $[t_1, t_2 - 1]$ gives the bound (5). For a network composed of a single source-terminal pair and a single link, this bound is tight.

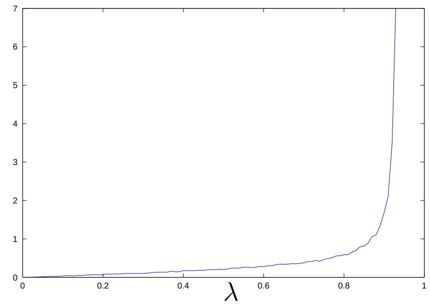

Fig. 3. Lower bound on achievable backlog for the simple multicast network with $q = 0.75$.

Using the bound (5),

$$Q(t_2) = \sum_{k=0}^{t_2-1} A(k) - \sum_{k=0}^{t_2-1} D(k)$$
$$\geq \sum_{k=t_1}^{t_2-1} \left(A(k) - \sum_{i \in F_c} \left(1 - \prod_{j:(i,j)\in E_c} \left(1 - S_{(i,j)}(k)\right) \right) \right)$$

Since this holds for all times t_1 and all cuts c,

$$Q(t_2) \geq \max_{c,t_1} \left\{ \sum_{k=t_1}^{t_2-1} \left(A(k) - \sum_{i \in F_c} \left(1 - \prod_{j:(i,j)\in E_c} \left(1 - S_{(i,j)}(k)\right) \right) \right) \right\}$$

In the case of Bernoulli arrivals and Bernoulli link connectivities, the argument of the maximum is simply a linear combination of binomial random variables. That is,

$$Z_n = \sum_{k=t-n}^{t-1} A(k)$$

is a binomial random variable with parameters n and λ. For each $i \in F_c$,

$$Y_{c,i,n} = \sum_{k=t-n}^{t-1} \left(1 - \prod_{j:(i,j)\in E_c} \left(1 - S_{(i,j)}(k)\right) \right).$$

is a binomial random variable with parameters n and $1 - \prod_{j:(i,j)\in E_c}(1 - q_{ij})$. Since the arrival and link connectivity processes are memoryless, the joint probability mass function of Z_n and all of the $Y_{c,i,n}$ is independent of t.

The steady-state expected backlog is lower bounded as

$$\liminf_{t\to\infty} \mathbf{E}[Q(t)] \geq \mathbf{E}\left[\sup_{n\geq 0, c\in\mathcal{C}} \left\{ Z_n - \sum_{i\in F_c} Y_{c,i,n} \right\} \right] \quad (6)$$

The random variables $Z_n - \sum_{i\in F_c} Y_{c,i,n}$ are fairly straightforward to analyze individually. However, the expectation of their supremum, particularly since these random variables are correlated, is significantly more difficult to analyze. As was shown in the previous section, we have obtained analytical results in a special case.

Here we will also show a simple network example where a lower bound can be computed numerically. Consider the network shown in Figure 2. A cut separating the source s from terminals t_1 and t_2 is also shown in the figure. In this example there are seven directed cuts separating the source from each of the terminals. For the case where $q_{ij} = 0.75$ for all edges, a curve of the expected steady-state value of this lower bound (computed by taking empirical averages) is shown in Figure 3. This figure gives a lower bound on the steady state expected number of packets queued in the network when packets enter the network at vertex s according to a Bernoulli processes with rate λ.

By applying Jensen's inequality to the right-hand side of (6), we establish the following lower bound on the expected steady-state queue length.

$$\liminf_{t\to\infty} \mathbf{E}[Q(t)] \geq$$
$$\sup_{n\geq 0, c\in\mathcal{C}} \left\{ \lambda n - \sum_{i\in F_c} \left(1 - \prod_{j:(i,j)\in E_c} (1-q_{ij}) \right) n \right\}$$

This lower bound relates the queue length to the minimum-cut capacity of the network. The lower bound can be made arbitrarily large, and hence the rate λ cannot be stably supported, if there exists some cut with

$$\lambda > \sum_{i\in F_c} \left(1 - \prod_{j:(i,j)\in E_c} (1-q_{ij}) \right).$$

This condition is exactly the necessity part of the capacity theorem for erasure networks [9].

C. Upper bound on queue length for random linear coding in the one-hop multicast network

Now we will analyze the queue length for the one-hop multicast problem under a simple random linear coding strategy that we call 'code over queue contents'. This strategy sequentially performs rounds of encoding, each of which lasts several time slots. When a round of encoding begins, all packets in the queue are selected. Let C be the total number of packets in the queue at the start of a round of encoding. Encoded packets formed from random linear combinations of these packets C are then sent to the receivers. Any arrivals to the queues during a round of encoding will not be considered until the next round of encoding. The round of encoding ends when all receivers can decode all C packets. These C packets are removed from their queue, then the next round of encoding begins.

Each encoded packet is a random linear combination of the C packets in the current coding block and is formed in the following way. A set of coefficients a_i for $i = 1, \ldots, C$ are randomly and uniformly chosen from the finite field $\mathbb{F}_d$ and these coefficients are used as weights in forming a linear combination using addition and multiplication over $\mathbb{F}_d$. The size of the finite field d is a power of two. A receiver can recover the original C packets once it has received C linearly independent encoded packets. The results presented below hold for any finite field $\mathbb{F}_d$ with $2 \leq d$. The time to transmit a coding block as well as the queue backlog are maximized for $d = 2$ and this case is used in obtaining our upper bound.

As described above, we assume the use of random linear codes, as introduced in [11]. For the one-hop multicast problem, digital fountain codes, such as Tornado codes introduced in [4], are also applicable and are known to be capacity-achieving in the limit of large block sizes. Our results on the queue length for packet coding do not directly apply when fountain codes are used; this is because we allow for a variable block size and the coding overhead that results for fountain codes with small block sizes is not captured in our result. However, similar techniques may be useful in analyzing the queue length performance for digital fountain codes.

Recall that $Q(t)$ denotes the number of packets in the queue at time t. The queue length process $Q(t)$ generally does not evolve as a Markov chain. This is because each round of encoding lasts several time slots, and the distribution of the length of a round of encoding is not memoryless. Although the process $Q(t)$ is not a Markov chain, the process $Q(t_0), Q(t_1), Q(t_2), \ldots$ is a Markov chain, where $t_0, t_1, t_2, \ldots$ are the starting times of successive rounds of encoding. Thus, we can apply tools for Markov chains to analyze the average value of the *embedded Markov chain* $Q(t_n)$. This will provide the steady-state average value of the queue backlog at the start of each round of encoding.

Before presenting Theorem 2, the main result of this section, we will state a general lemma that is used in its proof. A proof of this lemma can be found in [5].

Lemma 4: Let $X(t)$ be a Markov chain with countable state space $\mathcal{X}$. Let $r : \mathcal{X} \to \mathbb{R}$ be a cost associated with being in each state in $\mathcal{X}$, and let $h : \mathcal{X} \to \mathbb{R}_+$ be a nonnegative function on $\mathcal{X}$. Then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{E}[r(X(k))] &\le \\ \sup_{x\in\mathcal{X}} \{r(x) + \mathbf{E}[h(X(t+1)) \,|\, X(t) = x] - h(x)\}&.\end{aligned}$$

Theorem 2: The steady-state average of the embedded Markov chain $Q(t_k)$ satisfies

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mathbf{E}[Q(t_k)] &\le \\ 4\ln(M)\left(\frac{\lambda}{q-\lambda}\right)^2 + \left(8\sqrt{\ln(M)}+6\right)\frac{\lambda}{q-\lambda}&.\end{aligned}$$

Proof: Throughout this proof, we will let $\rho = \lambda/q$ denote *load factor* associated with the queue. To prove the upper bound, we will use the bound given in Lemma 4. Specifically, we will use the function

$$h(x) = \frac{2x}{1-\rho}.$$

By applying Lemma 4 we get

$$x + \mathbf{E}[h(Q(t_{i+1}) \,|\, Q(t_i) = x] - h(x) = x + \frac{2(\lambda \mathbf{E}[T(x)] - x)}{1-\rho}$$

where $\mathbf{E}[T(x)]$ denotes the expected time to transmit a coding block containing x packets. By Theorem 2 in [8],

$$\mathbf{E}[T(x)] \le \frac{1}{q}\left(x + 2\sqrt{(0.78x+3.37)\ln(M)} + 2.61\right).$$

Using this bound on $\mathbf{E}[T(x)]$, we get the upper bound

$$x + \mathbf{E}[h(Q(t_{i+1}) \,|\, Q(t_i) = x] - h(x) \le \frac{\rho(2\alpha(x)-1)-1}{1-\rho}x \tag{7}$$

where

$$\alpha(x) = \frac{x + 2\sqrt{(0.78x+3.37)\ln(M)} + 2.61}{x}.$$

The value of $x \ge 0$ that maximizes (7) is

$$x = 4\ln(M)\left(\frac{\rho}{1-\rho}\right)^2,$$

and the associated maximum is

$$4\ln(M)\left(\frac{\rho}{1-\rho}\right)^2 + \left(8\sqrt{\ln(M)}+6\right)\frac{\rho}{1-\rho}.$$

■

This theorem shows that the average queue length at departure times for the 'code over queue contents' strategy scales as $O(\ln(M))$.

IV. Conclusions

In this paper we considered a simple multicast model, where a single transmitter sends packets to M receivers over lossy links. The transmitter is equipped with a queue, and our goal was to find a transmission strategy that minimizes the expected number of packets in the queue. While finding the queue length minimizing strategy is still an open problem, here we found a lower bound on achievable performance and an upper bound on the performance for a random linear coding strategy. Specifically, we have shown that queue length must scale as $\Omega(\ln(M))$, and that queue length under the random linear coding strategy scales as $O(\ln(M))$. Hence, the random linear coding strategy is order-optimal with respect to the number of receivers.

In addition to our analysis for the one-hop multicast model, we provided a framework for analyzing multicast in more general erasure networks. This framework provides a method for lower bounding the minimum achievable total queue backlog throughout a network. Here we have shown an example where a lower bound on achievable backlog was computed numerically for a two-hop network.

Acknowledgment

The authors wish to thank the anonymous reviewers, whose helpful comments helped to improve the paper.

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